PMM U.S.S.R.,Vol.47,No.3,pp.414-419,1984
Printed in Great Britain

0021-8928/84 $10.00+0.00

UDC 539.374

ON FORMULATIONS OF THE PROBLEM OF THE THEORY OF PLASTICITY*

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Formulation of the problem of the theory of perfect plasticity includes the definition of a set of virtual (sample) velocity fields. In the definition of the latter for an incompressible medium we may or may not include the solenoidality requirement. Both variants result in different but connected statements of the problem. In this paper are presented and briefly considered statements on the quasi-static problem of perfectly plastic incompressible bodies. The equivalence of various formulations is established. This enables the selection for computation of this or that formulation of the problem, which may be important, owing to the computational nonequivalence in various situations.

Sample velocity fields and statically admissible stress fields. Let an incompressible medium occupy the region Ω in R^n $(n=2,3)$ and be subjected to mass forces of density $\mathbf{f}$ defined in Ω, and a surface load of density $\mathbf{q}$ defined on part S_q of the boundary of region Ω. Let also $S_v = \partial\Omega \setminus \bar{S}_q$, $S_q = \partial\Omega \setminus \bar{S}_v$, where the stroke denotes the closing in R^n, and the velocity is defined on S_v (e.g., S_v is fixed).

The stress fields are considered in Ω from space S, and it is subsequently assumed stress components are at least summable in the square on Ω. The statically admissible for the load $\{\mathbf{f}, \mathbf{q}\}$ is called the stress field $\boldsymbol{\sigma}$ from S which balances it; the conditions of equilibrium in region Ω and at its boundaries may be written in the form of an equation of the principle of virtual velocities

$$\int_\Omega \boldsymbol{\sigma}\cdot\mathbf{e}\,dx - \int_\Omega \mathbf{f}\mathbf{v}\,dx - \int_{S_q} \mathbf{q}\mathbf{v}\,ds = 0,\ \forall \mathbf{v}\in\mathbf{V}$$

$$e_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x^j} + \frac{\partial v_j}{\partial x^i}\right),\quad i, j = 1, 2, \ldots, n$$

where $\mathbf{V}$ is the set of virtual (sample) velocity fields and x^i are Cartesian coordinates in R^n.

Let $\boldsymbol{\Sigma}$ be a set of self-balancing (i.e. balancing the load $\mathbf{f}=0, \mathbf{q}=0$) of the stress fields.

$$\boldsymbol{\Sigma} = \boldsymbol{\Sigma}(\Omega, S_v) = (\mathbf{E}(\mathbf{V}))^\circ = \left\{\boldsymbol{\sigma}\in \mathrm{S}: \int_\Omega \boldsymbol{\sigma}\cdot\mathbf{e}\,dx = 0\quad \forall \mathbf{e}\in\mathbf{E}(\mathbf{V})\right\}$$

where $\mathbf{E}(\mathbf{V})$ is the aggregate of deformation rate fields that correspond to velocity fields from $\mathbf{V}$. If s is some stress field from S that balances the load $\{\mathbf{f}, \mathbf{q}\}$ then $\boldsymbol{\Sigma}+\mathrm{s}$ is the aggregate of all statically admissible stress fields.

For incompressible media two sets of sample velocity fields are used

$$\mathbf{V}^1 = \mathbf{V}^1(\Omega, S_v) = \{\mathbf{u}\in \mathbf{C}^\infty(\bar{\Omega}): \operatorname{div}\mathbf{u} = 0,\ \mathbf{u}|_{S_v} = 0\} \tag{1}$$

$$\mathbf{V}^2 = \mathbf{V}^2(\Omega, S_v) = \{\mathbf{u}\in \mathbf{C}^\infty(\bar{\Omega}): \mathbf{u}|_{S_v} = 0\}$$

To the self-balancing stress fields correspond the sets $\boldsymbol{\Sigma}^1 = (\mathbf{E}(\mathbf{V}^1))^\circ$, $\boldsymbol{\Sigma}^2 = (\mathbf{E}(\mathbf{V}^2))^\circ$. To the same sets $\boldsymbol{\Sigma}^1$, $\boldsymbol{\Sigma}^2$ results the use in the capacity of $\mathbf{V}$ for closing $\bar{\mathbf{V}}^1, \bar{\mathbf{V}}^2$ in $\mathbf{H}^1(\Omega)$ of sets $\mathbf{V}^1, \mathbf{V}^2$, respectively.

The following sets of velocity fields were considered in /2/:

$$\mathbf{U}^1 = \mathbf{U}^1(\Omega, S_v) = [\{\mathbf{u}\in \mathbf{C}^\infty(\bar{\Omega}): \operatorname{div}\mathbf{u} = 0,\ \ \rho(\operatorname{supp}\mathbf{u}, \bar{S}_v) > 0\}]_{\mathbf{H}^1(\Omega)}$$

$$\mathbf{U}^2 = \mathbf{U}^2(\Omega, S_v) = [\{\mathbf{u}\in \mathbf{C}^\infty(\bar{\Omega}): \rho(\operatorname{supp}\mathbf{u}, \bar{S}_v) > 0\}]_{\mathbf{H}^1(\Omega)}$$

$$\mathbf{W}^1 = \mathbf{W}^1(\Omega, S_v) = \{\mathbf{u}\in \mathbf{H}^1(\Omega): \operatorname{div}\mathbf{u} = 0,\ \mathbf{u}|_{S_v} = 0\}$$

$$\mathbf{W}^2 = \mathbf{W}^2(\Omega, S_v) = \{\mathbf{u}\in \mathbf{H}^1(\Omega): \mathbf{u}|_{S_v} = 0\}$$

where ρ is the distance in R^n, $\operatorname{supp}\mathbf{u}$ is the carrier of function $\mathbf{u}$, $[A]_{\mathbf{H}^1(\Omega)}$ is the closure of

*Prikl.Matem.Mekhan.,Vol.47,No.3,pp.490-496,1983

set A in $\mathbf{H}^1(\Omega)$ and the conditions for which the relations $\mathbf{U}^1 = \mathbf{W}^1, \mathbf{U}^2 = \mathbf{W}^2$. The results of /3/ were used and the respective consideration was carried out in the case of $\partial\Omega = S_v$. Subsequently we limit ourselves to the case in which the preceding equalities are valid. Note that then $\bar{\mathbf{V}}^1, \bar{\mathbf{V}}^2$ coincide, respectively, with $\mathbf{U}^1 = \mathbf{W}^1$ and $\mathbf{U}^2 = \mathbf{W}^2$ and, consequently, the sets of self-balancing stress fields Σ^1, Σ^2 do not change, if in the definition of $\mathbf{V}^1$ and $\mathbf{V}^2$ for condition $\mathbf{u}|_{S_v} = 0$ we substitute the condition for vanishing of $\mathbf{u}$ in certain neighborhood of the set $\bar{S}_v$ in $\bar{\Omega}$.

Evidently any σ from Σ^2 belongs to Σ^1. On the other hand, under certain conditions on Ω, S_v for any τ from Σ^1 a pressure field will be found $p \in L_2(\Omega)$ such that $\tau + pg$ (g is the metric tensor) belongs to Σ^2 /2/. Subsequently this property will be used for proving the equivalence of various statements of the problem of the theory of perfect plasticity.

Quasi-static problem of the theory of perfect plasticity. Let region Ω be filled by an incompressible, rigid-perfectly plastic medium. To each point x from Ω corresponds a cylinder C_x of admissible (not emerging beyond the yield surface) stresses in the space of second rank symmetric tensors. The yield surface is the boundary of that cylinder. Let $\sigma(x)$ be the stress at point x; we shall denote by indices d and s the deviatoric and spherical components of tensors. The $\sigma(x)$ is admissible, if $\sigma(x)$ belongs to set C_x or, what is the same, $\sigma^d(x)$ belongs to C_x^d, where C_x^d is the set that corresponds to cylinder C_x in the space of deviators of second rank symmetric tensors.

Since the body is not assumed homogeneous, hence the sets C_x^d for various points of the body are generally not the same. Let for all x from Ω the sets C_x^d be convex, closed, contain the set $\{\sigma : \sigma = \sigma^d, \sigma^d \cdot \sigma^d \leqslant r^2\}, r > 0$, and be contained in the set $\{\sigma : \sigma = \sigma^d, \sigma^d \cdot \sigma^d \leqslant R^2\}$. The mapping $x \rightarrow C_x^d$ is assumed measurable /4/.

For definiteness we shall consider the case when S_v is fixed (the case of nonuniform kinematic conditions on S_v is similarly considered, if the load $\{\mathbf{f}, \mathbf{q}\}$ is below the limit, if it is at the limit, the quasi-static problem may have no solution /5,6/ when the kinematic conditions are nonuniform).

Let $\mathbf{E}$ be the set of kinematically admissible deformation rates. It lies in space $\mathbf{F}$, and on $\mathrm{S} \times \mathbf{F}$ the bilinear form $\langle \cdot, \cdot \rangle$ is defined, which for summable $\sigma, \mathbf{e}$ from these spaces assumes the value

$$\langle \sigma, \mathbf{e} \rangle = \int_\Omega \sigma(x) \cdot \mathbf{e}(x)\, dx$$

Let A° be the polar line of set A (relative to the pair $\mathrm{S}, \mathbf{F}$) /7/. It is reasonable to assume that the relation $\mathbf{E}(\mathbf{V}) \subseteq \mathbf{E} \subseteq (\mathbf{E}(\mathbf{V}))^{\infty}$, where $\mathbf{V}$ is the set of sample velocity fields, is satisfied for $\mathbf{E}$. Then $\Sigma = (\mathbf{E}(\mathbf{V}))^\circ = \mathbf{E}^\circ$.

If s is some stress vector field from S that balances a given load $\{\mathbf{f}, \mathbf{q}\}$, then the quasi-static problem of the theory of perfect plasticity for the considered body consists of finding a pair $\sigma, \mathbf{e}$ that satisfies the conditions

$$\sigma \in \Sigma + \mathrm{s}, \quad \sigma \in C, \quad \mathbf{e} \in \mathbf{E}, \quad D(\mathbf{e}) = \langle \sigma, \mathbf{e} \rangle \tag{2}$$

where C is the set of stress fields from S which do not emerge beyond the yield surface

$$C = \{\sigma \in \mathrm{S} : \sigma^d(x) \in C_x^d \text{ for almost all } x \in \Omega\} \tag{3}$$

where D is the dissipation in the body, uniform of first power /1,8/ with respect to deformation rates.

The first of relations (2) indicate that the stress field σ balances the specified load, the second that the field σ does not emerge beyond the yield surface, and the third that the deformation rates $\mathbf{e}$ are kinematically admissible. The last of relations (2) shows the associated law in overall form /9/, and in it

$$D(\mathbf{e}) = \sup_{\tau \in C} \langle \tau, \mathbf{e} \rangle \tag{4}$$

Formulation in the form of a pair of extremal problems. With the load $\mathbf{s}$ (in the considered here problems s replaces the load $\{\mathbf{f}, \mathbf{q}\}$, and for brevity is subsequently called the load) and link the static (α_s) and kinematic (β_s) limit coefficients /9,10/

$$\alpha_s = \sup\{\mu : \mu \geqslant 0, \ \sigma - \mu \mathrm{s} \in \Sigma, \ \sigma \in C\}$$
$$\beta_s = \inf\{D(\mathbf{e}) : \mathbf{e} \in \mathbf{E}, \ \langle \mathrm{s}, \mathbf{e} \rangle = 1\} \tag{5}$$

In the theory of plasticity it is assumed that $\alpha_s = \beta_s$, and the conditions that ensure this equality were recently indicated /9,11,12/. The common value of α_s and β_s is denoted by n_s and called the load safety factor.

We call the problem (in stresses) of determining the extremum in (5) at which $\alpha_s, -H_s$ is achieved, and the (kinematic) problem of finding the extremum in (5) at which $\beta_s, -K_s$ is reached.

Problem (2) has no solution when $n_s < 1$, but when $n_s > 1$ it is solvable, and then $\mathbf{e} = 0$. When $n_s = 1$ the pair of extremals $(\boldsymbol{\sigma}, \mathbf{e})$ of problem H_s, K_s yields the solution of problem (2); conversely, if $(\boldsymbol{\sigma}, \mathbf{e})$ is the solution of problem (2), then $\boldsymbol{\sigma}$ and $\mathbf{e}$ are respectively, the extremals of H_s and K_s /9/. Thus the proof of equality $\alpha_s = \beta_s$ and the attainment of extremals in problems H_s, K_s are basic to the theory of the quasi-static of perfect plastic body.

The formulation of problems H_s, K_s comprises the definition of space $\mathrm{S}, \mathbf{F}$ of the set of kinematically admissible deformation rates $\mathbf{E}$ and of the set of sample velocity fields $\mathbf{V}$.

The $\mathrm{S}, \mathbf{F}$ spaces. We represent the second rank metric tensors $\mathbf{t}$ in the form $\mathbf{t} = (\mathbf{t}^d, \mathbf{t}^s)$, where

$$\mathbf{t}^d = \mathbf{t} - \frac{1}{n} t_k{}^k \mathbf{g}, \quad \mathbf{t}^s = \frac{1}{n} t_k{}^k \mathbf{g}$$

and the respective spaces S and $\mathbf{F}$ in the form $\mathrm{S}^d \times \mathrm{S}^s$ and $\mathbf{F}^d \times \mathbf{F}^s$.

If, as stated above, the yield surfaces for all points of the body are contained in certain Mises surfaces $\boldsymbol{\sigma}^d \cdot \boldsymbol{\sigma}^d = R^2$, i.e. the components of the stress tensor deviators are bounded, it is usual to assume that

$$\mathrm{S}^d = \mathbf{L}_\infty{}^d(\Omega) = \{\boldsymbol{\sigma} : \sigma_{ij} = \sigma_{ij}, \sigma_k{}^k = 0, \sigma_{ij}{}^d \in L_\infty(\Omega)\}$$
$$\|\boldsymbol{\sigma}\|_{\mathbf{L}_\infty{}^d(\Omega)} = \|\sqrt{\boldsymbol{\sigma} \cdot \boldsymbol{\sigma}}\|_{L_\infty(\Omega)}$$

Then assuming the duality of spaces S^d and $\mathbf{F}^d$, it is possible to set

$$\mathbf{F}^d = \mathbf{F}^d_{(1)} = \mathbf{L}_1{}^d(\Omega) = \{\mathbf{e} : e_{ij} = e_{ji}, e_k{}^k = 0, e_{ij} \in L_1(\Omega)\}$$
$$\|\mathbf{e}\|_{\mathbf{L}_1{}^d(\Omega)} = \|\sqrt{\mathbf{e} \cdot \mathbf{e}}\|_{L_1(\Omega)}$$

It can be shown that $\mathrm{S}^d = (\mathbf{F}^d_{(1)})'$. As usual X' represent the space of linear continuous functionals on space X. The duality of S^d and $\mathbf{F}^d$ is also ensured, if $\mathbf{F}^d = \mathbf{F}^d_{(2)} = (\mathbf{L}^d_\infty(\Omega))'$. The space $(\mathbf{L}^d_\infty(\Omega))'$ is formed by elements $\mathbf{e}$ for which $e_{ij} = e_{ji}$, $e_k{}^k = 0$, $e_{ij} \in L_\infty'(\Omega)$. The norm of an element $\mathbf{e}$ in this space is equivalent to the norm

$$\|\mathbf{e}\| = \Big(\sum_{i,j} \|e_{ij}\|^2_{L_\infty'(\Omega)}\Big)^{1/2}$$

As the S^s and $\mathbf{F}^s$ we use the space

$$\mathbf{L}_2{}^s(\Omega) = \Big\{\mathbf{t} : \mathbf{t} = \frac{1}{n} t_k{}^k \mathbf{g},\ t_k{}^k \in L_2(\Omega)\Big\}, \qquad \|\mathbf{t}\|_{\mathbf{L}_2{}^s(\Omega)} = \Big\|\frac{1}{n} t_k{}^k\Big\|_{L_2(\Omega)}$$

Then the selection of $\mathbf{F}^d = \mathbf{F}^d_{(1)}$ yields the pair of spaces

$$\mathrm{S} = \mathrm{S}^d \times \mathbf{L}_2{}^s, \quad \mathbf{F} = \mathbf{F}_{(1)} = \mathbf{F}^d_{(1)} \times \mathbf{L}_2{}^s$$

Then $\mathrm{S} = \mathbf{F}_{(1)}$ and for every $\boldsymbol{\sigma}$ from S and every $\mathbf{e}$ from $\mathbf{F}$ we have

$$\langle \boldsymbol{\sigma}, \mathbf{e} \rangle = \langle \boldsymbol{\sigma}^d, \mathbf{e}^d \rangle + \langle \boldsymbol{\sigma}^s, \mathbf{e}^s \rangle = \sigma_{ij}{}^d(e_{ij}{}^d) + \sigma_{ij}{}^s(e_{ij}{}^s) = \int_\Omega \boldsymbol{\sigma}(x) \cdot \mathbf{e}(x)\, dx$$

The selection of $\mathbf{F}^d = \mathbf{F}^d_{(2)}$ yields a pair of spaces

$$\mathrm{S} = \mathrm{S}^d \times \mathbf{L}_2{}^s, \quad \mathbf{F} = \mathbf{F}_{(2)} = \mathbf{F}^d_{(2)} \times \mathbf{L}_2{}^s$$

then $\mathbf{F}_{(2)} = \mathrm{S}'$ and for every $\boldsymbol{\sigma}$ from S and every $\mathbf{e}$ from $\mathbf{F}$ we have

$$\langle \boldsymbol{\sigma}, \mathbf{e} \rangle = \langle \boldsymbol{\sigma}^d, \mathbf{e}^d \rangle + \langle \boldsymbol{\sigma}^s, \mathbf{e}^s \rangle = e_{ij}{}^d(\sigma_{ij}{}^d) + e_{ij}{}^s(\sigma_{ij}{}^s)$$

The sets $\mathbf{V}, \mathbf{E}, \boldsymbol{\Sigma}$. We use $\mathbf{V}^1$ or $\mathbf{V}^2$ as sets of sample velocity fields, and respectively $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}^1 = (\mathbf{E}(\mathbf{V}^1))^\circ$ or $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}^2 = (\mathbf{E}(\mathbf{V}^2))^\circ$; as noted above, it is possible to use in the capacity of $\mathbf{V}$ some other sets, without changing $\boldsymbol{\Sigma}^1$ and $\boldsymbol{\Sigma}^2$.

The aim of obtaining extreme values in (5), i.e. the solvability of the quasi-static problem of the theory of perfect plasticity, compels us to widen the sets $\mathbf{E}(\mathbf{V})$. We take as set $\mathbf{E}$ the kinematically admissible deformation rate appearing in the statement of problem (5) and the suitable closing sets $\mathbf{E}(\mathbf{V}^1)$, $\mathbf{E}(\mathbf{V}^2)$. We have, namely, the following four possibilities:

$$\mathbf{E} = \mathbf{E}_1{}^1 = [\mathbf{E}(\mathbf{V}^1)]_{(1)}, \quad \mathbf{E} = \mathbf{E}_2{}^1 = [\mathbf{E}(\mathbf{V}^1)]_{(2)}$$
$$\mathbf{E} = \mathbf{E}_1{}^2 = [\mathbf{E}(\mathbf{V}^2)]_{(1)}, \quad \mathbf{E} = \mathbf{E}_2{}^2 = [\mathbf{E}(\mathbf{V}^2)]_{(2)}$$

where brackets with subscripts (1) denote closure in $\mathbf{F}_{(1)}$ and those with subscript (2) in weak * topology of $\mathbf{F}_{(2)}$.

Note that since the polar line of sets $\mathbf{E}(\mathbf{V}^1)$, $\mathbf{E}(\mathbf{V}^2)$ coincides with the polar lines of their closure, hence

$$\Sigma^1 = (\mathbf{E}(\mathbf{V}^1))^\circ = (\mathbf{E}_1{}^1)^\circ = (\mathbf{E}_2{}^1)^\circ, \quad \Sigma^2 = (\mathbf{E}(\mathbf{V}^2))^\circ = (\mathbf{E}_1{}^2)^\circ = (\mathbf{E}_2{}^2)^\circ$$

Variants of the statement of quasi-static problem of the theory of perfect plasticity. In conformity with the possibility of different selection of spaces $\mathbf{S}, \mathbf{F}$ and sets $\mathbf{V}, \mathbf{E}$ we have four pairs of problems (5) enumerated by the sets $i, j = 1, 2$

$$\begin{aligned} &\sup\{\mu : \mu \geqslant 0, \sigma - \mu s \in \Sigma^j, \sigma \in C\} = \alpha^j \\ &\inf\{D(\mathbf{e}) : \mathbf{e} \in \mathbf{E}_i{}^j, \quad \langle s, \mathbf{e}\rangle = 1\} = \beta_i{}^j \end{aligned} \tag{6}$$

where the superscript s at α and β indicating the considered load is omitted. The problems of finding extremals in (6) are denoted, respectively, by H^j, $K_i{}^j$ with the subscript s again omitted.

The problems H^1 and $K_i{}^1$ $(i = 1, 2)$ may be formulated so that only deviatoric components of tensors figure in them. Indeed, since $\mathbf{e}^s = 0$ for every $\mathbf{e}$ from $\mathbf{E}_i{}^1$, hence it is possible to introduce such sets $\mathbf{E}_i{}^{1d}$ in $\mathbf{F}_{(i)}{}^d$ that

$$\mathbf{E}_i{}^1 = \mathbf{E}_i{}^{1d} \times \{0\}, \; i = 1, 2$$

which are obviously closed in $\mathbf{F}_{(1)}{}^d$ and in the weak * topology of $\mathbf{F}_{(2)}{}^d$, respectively. Then the set of self-balancing stress fields can be represented in the form $\Sigma^1 = \Sigma^{1d} \times \mathbf{L}_2{}^s$. Here Σ^{1d} lies in $\mathbf{S}^d$, and it is evident that $\Sigma^{1d} = (\mathbf{E}_1{}^{1d})^\circ = (\mathbf{E}_2{}^{1d})^\circ$. Finally, $C = C^d \times \mathbf{L}_2{}^s$, where C^d lies in $\mathbf{S}^d$, and it is possible to introduce function D^d on $\mathbf{F}_{(i)}{}^d$

$$D^d(\mathbf{e}^d) = D((\mathbf{e}^d, 0)) = \sup_{\tau \in C} \langle \tau^d, \mathbf{e}^d\rangle = \sup_{\sigma^d \in C^d} \langle \sigma^d, \mathbf{e}^d\rangle, \quad \mathbf{e}^d \in \mathbf{F}^d_{(i)}$$

Then obviously

$$\begin{aligned} &\alpha^1 = \sup\{\mu : \mu \geqslant 0, \sigma^d - \mu s^d \in \Sigma^{1d}, \sigma^d \in C^d\} \\ &\beta_i{}^1 = \inf\{D^d(\mathbf{e}^d) : \mathbf{e}^d \in \mathbf{E}_i^{1d}, \langle s^d, \mathbf{e}^d\rangle = 1\}, \quad i = 1, 2 \end{aligned} \tag{7}$$

In this case, if $\sigma(\mathbf{e})$ is the extremal of problem $H^1(K_i{}^1)$, then $\sigma^d(\mathbf{e}^d)$ is the extremal of respective problem (7). And conversely, if σ^d is the extremal of problem (7) in stresses, then $\sigma^d + pg$ for any p from $L_2(\Omega)$ is the extremal of problem H^1; if $\mathbf{e}^d$ is the extremal of the kinetic problem (7), then $e = e^d$ is the extremal of problem $K_i{}^1$.

The duality of problem $\mathbf{H}, \mathbf{K}$ and the attainment of extrema. Two extremal problems (in one of which the upper and in the other the lower bound are sought) linked in a definite manner are usually called dual /1/. In such problems the upper bound does never exceed the lower bound of the dual problem. The pair of problems (6) and (7) are in this sense dual, and from this follows that $\alpha^j \leqslant \beta_i{}^j$ $(i, j = 1, 2)$. This inequality expresses the known fact in the theory of plasticity that the static limit load factor is not higher than the kinematic.

Sometimes only two such problems which are dual in the above sense and whose upper and lower bounds are the same are called dual. Subsequently, the duality is understood exactly as follows: for the problem considered duality means that the static and kinematic limit load factors are the same.

Problems H^1 and $K_1{}^1$ are dual /9,11/. Since $\beta_1{}^1 \geqslant \beta_2{}^1$ (which obviously follows from the embedding of $\mathbf{E}_1{}^1$ in $\mathbf{E}_2{}^1$) and $\alpha^1 \leqslant \beta_2{}^1$, hence $\beta_1{}^1 = \alpha^1 = \beta_2{}^1$. The duality of problems H^2, $K_2{}^2$ is also established /9,12/: $\alpha^2 = \beta_2{}^2$.

Extremes in kinematic problems $K_1{}^1, K_1{}^2$ with sets of kinematically admissible deformation rates $\mathbf{E}_1{}^1, \mathbf{E}_1{}^2$ (i.e. in space $\mathbf{F}_{(1)}$) may not be attained. In other words, kinematic problems in space $\mathbf{F}_{(1)}$ are unsolvable. An example of the problem with smooth data, which has no solution in $\mathbf{F}_{(1)}$ is given in /12/. In space $\mathbf{F}_{(2)}$ kinematic problems are solvable. /9,14/.

In problems in stresses H^1 and H^2 the extremes are attainable /9,11,15/. In other words, the problems in stresses are solvable in S.

The conditions that ensure the duality of problems H^1 and $K_1{}^1$ as well as those ensuring the duality of problems H^2 and $K_2{}^2$ generally separate different classes of plastic bodies. The differences concern the character of yield surface dependence, or what is the same, of the set $C_x{}^d$ on point x of the body. For the (very wide) class of bodies described above these or other conditions are satisfied. In that case a natural question arises about the relation of α^1 and α^2 of problem H^1 to problem H^2, and of problem $K_i{}^1$ to problem $K_i{}^2$ $(i = 1, 2)$.

The equivalence of various statements of the problem of the theory of perfect plasticity. In the case of fixed boundary $(\partial\Omega = S_v)$ the respective problems are equivalent and $\beta_1{}^1 = \alpha^1 = \beta_2{}^1 = \beta_1{}^2 = \alpha^2 = \beta_2{}^2$ /12/. A similar statement can be obtained also in the case of mixed boundary conditions using data from /2/.

Theorem. Let a perfectly plastic medium of the considered type fill the bounded region Ω of class C^1. Let S_q be nonempty and $\mathbf{W}^1(\Omega, S_v) = \overline{\mathbf{V}}^1(\Omega, S_v)$.

Then: 1) problems H^1 and H^2 are equivalent in the sense that $\alpha^1 = \alpha^2$, if on σ the maximum is reached in problem H^2, then maximum σ is reached also in problem H^1, and of maximum is reached on τ in problem H^1, then a pressure field p from $L_2(\Omega)$ will be found such that on $\tau + pg$ a maximum is reached in problem H^2; and 2) the functional D whose lower bound is sought in problem K_i^2 $(i = 1, 2)$ assumes the value $+\infty$ on elements $\mathbf{E}_i^2$ that lie outside the set $\mathbf{E}_i^1$, hence the problems K_i^1 and K_i^2 coincide.

Remarks. 1°. The sufficient conditions of congruence of $W^1(\Omega, S_v)$ and $\bar{V}^1(\Omega, S_v)$ are defined by Theorems 3.1 and 3.2 in /2/.

2°. The pressure field p is uniquely defined by the deviatoric component τ. More exactly, if τ_1 and τ_2 belong to Σ^1, their deviatoric components coincide, and $\sigma_1 = \tau_1 + p_1\mathbf{g}$, $\sigma_2 = \tau_2 + p_2\mathbf{g}$ belongs to Σ^2, then obviously, $\sigma_1 = \sigma_2$ when $S_q \neq \varnothing$ and $\sigma_1 - \sigma_2 = c\mathbf{g}$, where c is an arbitrary constant, when $S_q = \varnothing$.

Proof. If $\sigma - \mu s$ for some $\mu \geqslant 0$ and σ from C belongs to Σ^2, then obviously, $\sigma - \mu s$ belongs to Σ^1, and conversely, when $\tau - \mu s$ for some $\mu \geqslant 0$ and τ from C belongs to Σ^1, then by the theorem 5.2 of /2/ a pressure field p from $L_2(\Omega)$ will be found such that $\tau - \mu s - pg$ belongs to Σ^2. Then evidently together with τ also $\tau + pg$ belongs to C. This proves the first statement of the theorem.

Note that $D(\mathbf{e}) = +\infty$ for $\mathbf{e}$ when $\mathbf{e}^s \neq 0$. Hence problem K_i^2 coincides with the similar problem in which $\mathbf{E}_i^2$ is exchanged for its intersection with the set $\mathbf{F}_{(i)}^d \times \{0\}$; that intersection is subsequently denoted by $\mathbf{E}_{i,0}^2$. Thus for proving the second assertion of the theorem it is sufficient to check that $\mathbf{E}_i^1 = \mathbf{E}_{i,0}^2$.

In turn, the last equality is equivalent to relation $(\mathbf{E}_i^1)^\circ = (\mathbf{E}_{i,0}^2)^\circ$. Indeed, the sets $\mathbf{E}_i^1$, $\mathbf{E}_{i,0}^2$ are convex, balanced and closed (when $i = 1$ in $\mathbf{F}_{(1)}$ and when $i = 2$ in the weak * topology of $\mathbf{F}_{(2)}$). Then by the theorem about the bipolar line /7/ $\mathbf{E}_i^1 = (\mathbf{E}_i^1)^{\circ\circ}$, $\mathbf{E}_{i,0}^2 = (\mathbf{E}_{i,0}^2)^{\circ,\circ}$ wherefrom follows the idicated equivalence.

Since $\mathbf{E}_i^1$ is contained in $\mathbf{E}_{i,0}^2$, then $(\mathbf{E}_i^1)^\circ$ contains $(\mathbf{E}_{i,0}^2)^\circ$, and it remains to verify that $(\mathbf{E}_i^1)^\circ$ is embeded in $(\mathbf{E}_{i,0}^2)^\circ$. Let τ belong to $(\mathbf{E}_i^1)^\circ = \Sigma^1$, then by Theorem 5.2 in /2/ a p from $L_2(\Omega)$ will be found such that $\tau + pg$ belongs to $\Sigma^2 = (\mathbf{E}_i^2)^\circ$. On the other hand, $-pg$ belongs to $(\mathbf{F}_{(i)}^d \times \{0\})^\circ$ and then evidently

$$\tau = (\tau + pg) + (-pg) \in (\mathbf{E}_i^2 \cap (\mathbf{F}_{(i)}^d \times \{0\}))^\circ = (\mathbf{E}_{i,0}^2)^\circ$$

which completes the proof

Corollary. Under the theorem conditions the following relations are valid:

$$n = \beta_1^1 = \alpha^1 = \beta_2^1 = \beta_1^2 = \alpha^2 = \beta_2^2$$

Thus for calculating the load margine coefficient (when conditions of the theorem are satisfied) it is possible to use either problem (6) or (7).

Note that the functional D is convex and positively uniform of first power and, by virtue of condition $\sigma^d \cdot \sigma^d \leqslant R^2$, for the considered class of media $D^d(\mathbf{e}) \leqslant R\|\mathbf{e}\|$ on $\mathbf{F}_{(1)}^d$. Then for determining its lower bound $\beta_1^1 = n$ the set $\mathbf{E}_1^1$ can be replaced everywhere by the set $\mathbf{E}(\mathbf{V}^1)$ that is dense in it or, for instance, $\mathbf{E}(\bar{\mathbf{V}}^1)$). In other words, it is always possible to use the sequence of smooth velocity fields as the minimizing sequence in the kinematic problem.

Finally, if the load margin coefficients determined or estimated using the extremal prc blem in stresses, the spherical component of the utilized stress fields can be excluded from consideration. Since $n = \alpha^1$, it is possible to use Σ^1 as the set of self-balancing stress fields. Since together with any stress fields $\tau = (\tau^d, \tau^s)$ belonging to Σ^1 and C, and the stress field $\tau' = (\tau^d, 0)$ belongs to these sets, hence in problem H^1 we can limit ourselves, for instance, to stress fields with zero spherical components. On the other hand, if it is necessary to determine the true stress (for, say, determining the stresses in technological problems of the theory of placticity), it is necessary to consider the problem in complete statement of H^2.

REFERENCES

1. SEDOV L.I., Mechanics of a Continous Medium, Vol.2, Moscow, NAUKA, 1970.
2. KAMENIARZH Ia.A., Statically admissible stress fields in incompressible media. PMM, Vol. 47, No.2, 1983.
3. LADYZHENSKAIA O.A., On certain problems of vector analysis and the generalized statement for the Navier—Stokes equations. Zap. Nauchn. Seminarov LOMI AN SSSR, Vol.59, No.9,1976.
4. IOFFE I.D. and TIKHOMIROV V.M., The Theory of Extremal Problems. Moscow, NAUKA, 1974.
5. KAMENIARZH Ia.A., On a class of problems of perfect plasticity with nonuniform kinematic boundary conditions. Dokl. AN SSSR, Vol.255, No.1, 1980.

6. KAMENIARZH Ia.A., On a class of problems of perfect plasticity with nonuniform kinematic boundary conditions (the case of limit load). Dokl. AN SSSR, Vol.257, No.5, 1981.
7. EDWARDS R., Functional Analysis. Moscow, MIR, 1969.
8. IVLEV D.D., On the dissipation function in theory of plastic media. Dokl. AN SSSR, Vol. 176, No.5, 1967.
9. NAYROLES B., Essai de théorie fonctionnelles structures rigides plastiques parfaites. J. Mec. Vol.9, No.3, 1970.
10. KOITER W.T., General Theorems of the Theory of Elastioplastic Media. Moscow, Izd. Inostr. Lit., 1961.
11. MOSOLOV P.P., On certain mathematical questions of the theory of incompressible visco-plastic media. PMM, Vol.42, No.4, 1978.
12. KAMENIARZH Ia.A., On dual problems of the theory of limit load for perfectly plastic bodies. Dokl. AN SSSR, Vol.245, No.1, 1979.
13. GOL'SHTEIN E.G., The Theory of Duality in Mathematical Programming and Its Applications, Moscow, NAUKA, 1971.
14. MOSOLOV P.P., On the problem of minimum of a functional. Izv. AN SSSR, Vol.31, No.6, 1967.
15. KAMENIARZH Ia.A., The plastic body under the action of a longitudinal load: the stress field. Dokl. AN SSSR, Vol.241, No.1, 1978.

Translated by J.J.D.